# Randomness in Number Theory

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Mahler Lectures 2011

| Number Theory | Probability Theory |
| --- | --- |
| Whole numbers | Random objects |
| Prime numbers | Points in space |
| Arithmetic operations | Geometries |
| Diophantine equations | Matrices |
| $\vdots$ | Polynomials |
| $\vdots$ | Walks |
| $\vdots$ | Groups |
| $\vdots$ | $\vdots$ |
| Automorphic forms | Percolation theory |

Dichotomy: Either there is a rigid structure (e.g. a simple closed formula) in a given problem, or the answer is difficult to determine and in that case it is random according to some probabilistic law.

- The probabilistic law can be quite unexpected and telling.
- Establishing the law can be very difficult and is often the central issue.

The randomness principle has implications in both directions.
$\Rightarrow$ Understanding and proving the law allows for a complete understanding of a phenomenon.
$\Leftarrow$ The fact that a very explicit arithmetical problem behaves randomly is of great practical value.

Examples:

- To produce pseudo-random numbers,
- Construction of optimally efficient error correcting codes and communication networks,
- Efficient derandomization of probablistic algorithms "expanders".

Illustrate the Dichotomy with Examples

(0) Is $\pi = 3.14159265358979323\ldots$ a normal number?
$\pi$ is far from rational;
Mahler (1953): $$\left|\pi - \frac{p}{q}\right| > q^{-42}.$$

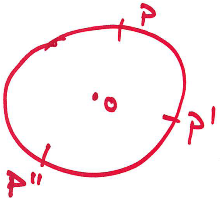
(1) In diophantine equations:
A bold conjecture: Bombieri–Lang takes the dichotomy much further. If $V$ is a system of polynomial equations with rational number coefficients ("a smooth projective variety defined over $\mathbb{Q}$"), then all but finitely many rational solutions arise from ways that we know how to make them (parametric, special subvarieties, group laws ...)
"The ignorance conjecture"

(2) A classical diophantine equation

Sums of three squares: for $n > 0$, solve

$$x^2 + y^2 + z^2 = n; \qquad x, y, z \in \mathbb{Z}.$$

If $P = (x, y, z)$, $d^2(P, 0) = n$.




$\mathcal{E}(n) :=$ set of solutions.

e.g. for $n = 5$, the $P$'s are

$$(\pm 2, \pm 1, 0), (\pm 1, \pm 2, 0), (\pm 2, 0, \pm 1),$$
$$(\pm 1, 0, \pm 2), (0, \pm 2, \pm 1), (0, \pm 1, \pm 2)$$

$N(n) := \#\mathcal{E}(n)$, the number of solutions, so $N(5) = 24$.

$N(n)$ is not a random function of $n$ but it is difficult to understand.

Gauss/Legendre (1800): $N(n) > 0$ iff $n \neq 4^a(8b+7)$.
(This is a beautiful example of a local to global principle.)

$N(n) \approx \sqrt{n}$ (if not zero).
Project these points onto the unit sphere

$$P = (x, y, z) \mapsto \frac{1}{\sqrt{n}}(x, y, z) \in S^2.$$

We have no obvious formula for locating the $P$'s and hence according to the dichotomy they should behave randomly. It is found that they behave like $N$ randomly placed points on $S^2$.

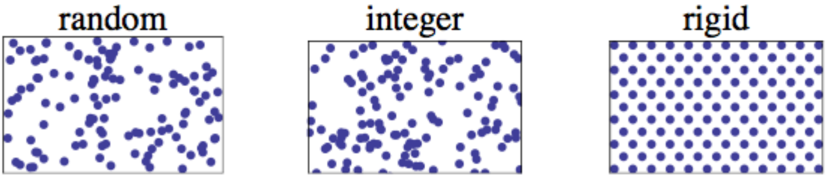



Figure 1. Lattice points coming from the prime $n = 1299709$ (center), versus random points (left) and rigid points (right). The plot displays an area containing about 120 points.

- One can prove some of these random features.
- It is only in dimension 3 that the $\widehat{\mathcal{E}}(n)$'s are random. For dimensions 4 and higher, the distances between points in $\widehat{\mathcal{E}}(n)$ have 'explicit' high multiplicities. For 2 dimensions there aren't enough points on a circle — not random.

(3) Examples from Arithmetic:

$P$ a (large) prime number. Do arithmetic in the integers keeping only the remainders when divided by $p$. This makes $\{0, 1, \ldots, p-1\} := \mathbb{F}_p$ into a finite field.

Now consider $x = 1, 2, 3, \ldots, p-1$ advancing linearly.
How do $\overline{x} := x^{-1} \pmod{p}$ arrange themselves?
Except for the first few, there is no obvious rule, so perhaps randomly?

Experiments show that this is so. For example, statistically, one finds that $x \mapsto \overline{x}$ behaves like a random involution of $\{1, 2, \ldots, p-1\}$.

One of the many measures of the randomness is the sum

$$S(1, p) = \sum_{x=1}^{p-1} e^{2\pi i (x+\overline{x})/p}.$$

If random, this sum of $p-1$ complex numbers of modulus 1 should cancel to about size $\sqrt{p}$.

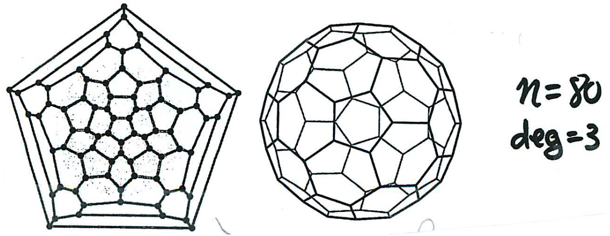
Fact: $|S(1,p)| \leq 2\sqrt{p}$. (A. Weil 1948)

Follows from the "Riemann hypothesis for curves over finite fields".

The fact that arithmetic operations such as $x \mapsto \overline{x} \pmod{p}$ are random are at the source of many pseudo-random constructions: e.g.

Ramanujan Graphs:

These are explicit and optimally highly connected sparse graphs (optimal expanders).




Largest known planar cubic Ramanujan graphs

Arithmetic construction:
$q \equiv 1 \pmod{20}$ prime

$$1 \le i \le q-1 \quad ; \quad i^2 \equiv -1 \pmod{q}$$
$$1 \le \beta \le q-1 \quad ; \quad \beta^2 \equiv 5 \pmod{q}$$

$S$ the six $2 \times 2$ matrices with entries in $\mathbb{F}_q$ and of determinant 1.

$$S = \left\{ \frac{1}{\beta} \begin{bmatrix} 1 \pm 2i & 0 \\ 0 & 1 \mp 2i \end{bmatrix}, \frac{1}{\beta} \begin{bmatrix} 1 & \pm 2 \\ \mp 2 & 1 \end{bmatrix}, \frac{1}{\beta} \begin{bmatrix} 1 & \pm 2i \\ \pm 2i & 1 \end{bmatrix} \right\}$$

Let $V_q$ be the graph whose vertices are the matrices $A \in \mathrm{SL}_2(\mathbb{F}_q)$, $|V_q| \sim q^3$, and edges run between $g$ and $sg$ with $s \in S$ and $g \in V_q$.

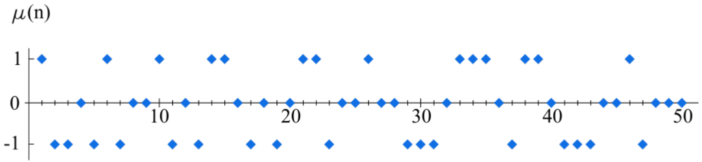
$V_q$ is optimally highly connected, 6 regular graph on $|\mathrm{SL}_2(\mathbb{F}_q)|$ vertices, optimal expander. Here arithmetic mimics or even betters random.

(4) The Möbius Function

$$n \geq 1, \quad n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

$$\mu(n) = \begin{cases} 0 & \text{if } e_j \geq 2 \text{ for some } j, \\ (-1)^k & \text{otherwise.} \end{cases}$$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mu(n)$ | 1 | −1 | −1 | 0 | −1 | 1 | −1 | 0 | 0 | 1 |

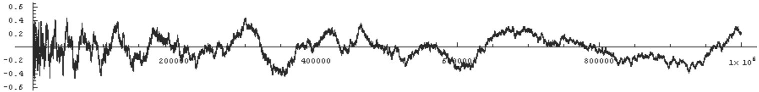



Is $\mu(n)$ random? What laws does it follow.
There is some structure, e.g. from the squares

$$\mu(4k) = 0 \qquad \text{etc.}$$

One can capture the precise structure/randomness of $\mu(n)$ via dynamical systems, entropy, ....
Very simplest question thinking of a random walk on $\mathbb{Z}$ moving to the right by 1 if $\mu(n) = 1$, to the left if $\mu(n) = -1$, and sticking if $\mu(n) = 0$. After $N$ steps?

$$\frac{1}{N}\sum_{n\le N}\mu(n), \qquad N \le 100\,000$$




Is

$$\left|\sum_{n\le N}\mu(n)\right| \ll_\varepsilon N^{1/2+\varepsilon}, \quad \varepsilon > 0?$$

This equivalent to the Riemann hypothesis! So in this case establishing randomness is one of the central unsolved problems in mathematics.

One can show that for any $A$ fixed and $N$ large,

$$\left|\sum_{n\le N}\mu(n)\right| \le \frac{N}{(\log N)^A}.$$

(5) The Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad s > 1$$

it is a complex analytic function of $s$ (all $s$).

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Riemann Hypothesis: All the nontrivial zeros $\rho$ of $\zeta(s)$ have real part $1/2$. Write $\rho = 1/2 + i\gamma$ for the zeros.

$$\gamma_1 = 14.21\ldots \qquad \text{(Riemann)}$$

and the first $10^{10}$ zeros are known to satisfy RH.

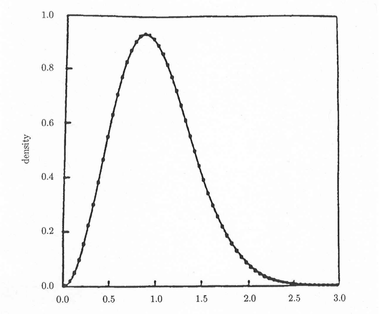
$$0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \ldots$$

Are the $\gamma_j$'s random?

Scale first so as to form meaningful local statistics

$$\widehat{\gamma}_j := \frac{\gamma_j \log \gamma_j}{2\pi}$$

$\widehat{\gamma}_j$, $j = 1, 2, \ldots$ don't behave like random numbers but rather like eigenvalues of a random (large) hermitian matrix! GUE




Nearest neighbor spacings among 70 million zeroes beyond the $10^{20}$-th zero of zeta, versus $\mu_1(\mathrm{GUE})$

(6) Modular Forms

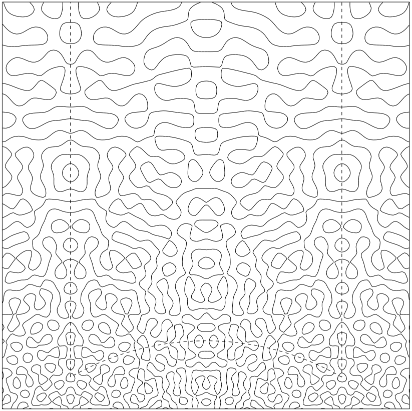
Modular (or automorphic) forms are a goldmine and are at the center of modern number theory. I would like to see an article "The Unreasonable Effectiveness of Modular Forms"
Who so? I think it is because they violate our basic principle.

- They have many rigid and many random features.
- They cannot be written down explicitly (in general)
- But one can calculate things associated with them to the bitter end, sometimes enough to mine precious information.

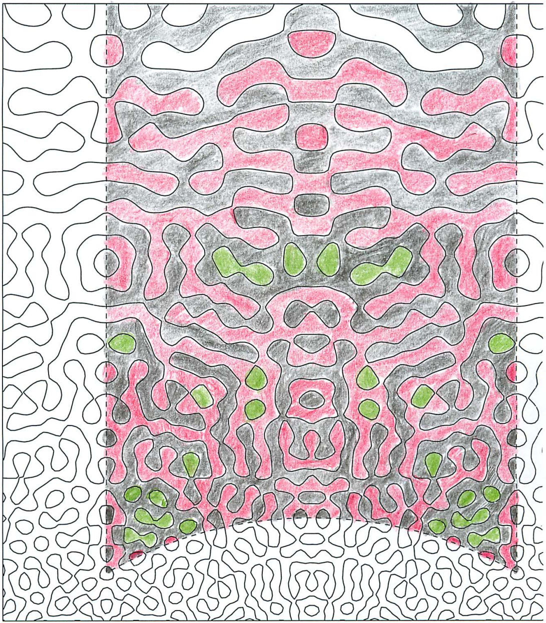
Below is the nodal set $\{\phi = 0\}$ of a highly excited modular form for $\mathrm{SL}_2(\mathbb{Z})$.

$$\Delta\phi + \lambda\phi = 0, \quad \lambda = \frac{1}{4} + R^2.$$

$\phi(z)$ is $\mathrm{SL}_2(\mathbb{Z})$ periodic. Is the zero behaving randomly? How many components does it have?



Hejhal–Rackner nodal lines for $\lambda = 1/4 + R^2$, $R = 125.313840$



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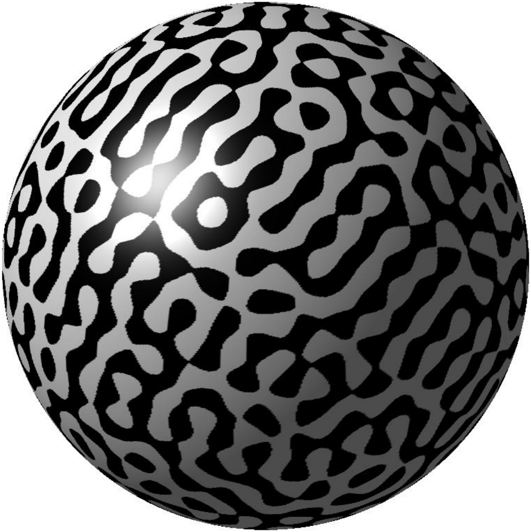


The physicists Bogomolny and Schmit (2002) suggest that for random waves

$$N(\phi_n) = \#\text{ of components} \sim cn$$

$c = \dfrac{3\sqrt{3} - 5}{\pi}$, comes from an exactly solvable critical percolation model!

- The modular forms apparently obey this rule. Some of this but much less can be proven.
- These nodal lines behave like random curves of degree $\sqrt{n}$.

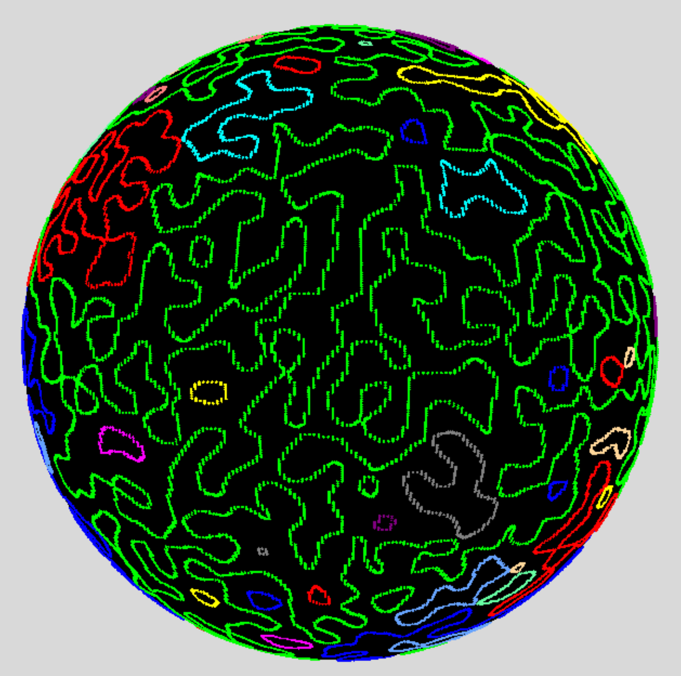
(7) Randomness and Algebra?

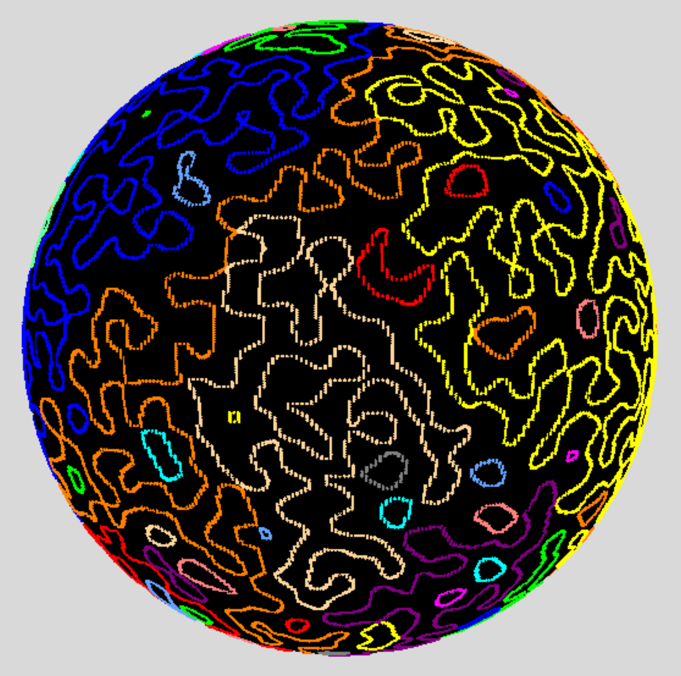
How many ovals does a random real plane projective curve of degree $t$ have?

Harnack: $\#$ of ovals $\leq \frac{(t-1)(t-2)}{2} + 1$

Answer: the random curve is about 4% Harnack,

$\#$ of ovals $\sim c' t^2$, $c' = 0.0182\ldots$ (Nazarov–Sodin, Nastasescu).





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